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Universal solutions for the streamwise variation of the temperature of a moving sheet in the presence of a moving fluid

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Abstract

A method has been developed for determining the streamwise variation of the temperature of a moving sheet in the presence of a co-flowing fluid. The solution does not depend on any material property of the sheet, its velocity, or its thickness. The solution is also independent of the properties of the fluid aside from the Prandtl number. Furthermore, the actual velocities of the sheet and the fluid need not be specified, but only their ratio is required. In the development of the method, a large knowledge base was first created by solving the differential equations for mass, momentum, and energy. The tabulated knowledge base served as input to a purely algebraic procedure whose end result is the streamwise variation of the sheet temperature. The procedure is iterative but requires no more than a least-squares curve-fitting capability. The iterative procedure is robust in that the converged result is independent of the initial iterant. It is also self correcting in the presence of an inadvertent error. Another method for determining the streamwise temperature variation, the relative-velocity model, was also investigated, and its accuracy assessed.

Keywords: Convective heat transfer; Moving sheet; Universal solution; Flat-plate boundary layer

1. Introduction

The thermal processing of sheet-like materials is a necessary operation in the production of paper, linoleum, polymeric sheets, roofing shingles, insulating materials, and fine-fiber matts. In virtually all such processing operations, the sheet moves parallel to its own plane. The moving sheet may induce motion in the neighboring fluid or, alternatively, the fluid may have an independent forced-convection motion that is parallel to that of the sheet. Representative applications involving a moving sheet and an independently moving fluid are illustrated in Figs. 1 and 2.

Heat transfer between the sheet and the adjacent fluid is usually initiated at the first contact between the media. If the fluid temperature is lower than that of the sheet, the sheet temperature decreases in the streamwise direction. Alternatively, if the fluid temperature is greater than the sheet temperature, there is a streamwise increase of the latter. The design of a thermal processing station for moving sheets requires a knowledge of heat

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Nomenclature

a_0, a_1, a_2	constants defined in Eq. (7)
b_0, b_1, b_2	constants defined in Eq. (27)
c_0, c_1, c_2	constants defined in Eq. (31)
c_s	heat capacity of the moving sheet
f	dimensionless stream function
k	thermal conductivity
$\dot{m}$	mass flow rate of the moving sheet
Pr	Prandtl number
q_w	local heat flux into the sheet
Re_x	Reynolds number based on the streamwise coordinate
U	velocity
u, v	local velocity components in the x- and y-direction
t	thickness of the moving sheet
W	width of the moving sheet
x, y	coordinates

Greek symbols

α	thermal diffusivity
η	similarity variable defined in Eq. (1)
Θ_0, Θ_1, Θ_2	dimensionless sheet temperature functions
ρ	density
λ	variable of integration
μ	dynamic viscosity
ν	kinematic viscosity
τ_{wall}	wall shear stress
χ	dimensionless streamwise coordinate defined in Eq. (26)

Subscripts

f	fluid
rel	relative
RVM	relative-velocity model
s	sheet
∞	freestream

Superscript

Roman numerals	iteration number

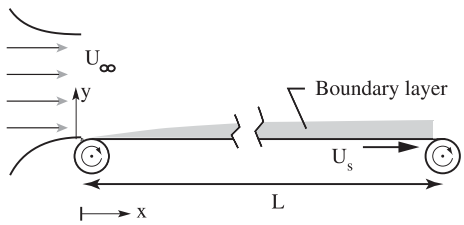

Fig. 1. A processing station where a ducted flow is passed over a moving sheet.

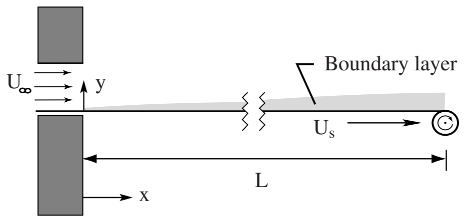

Fig. 2. A processing station where an extruded sheet-like material is washed by a fluid emerging through a slot in the die which produces the sheet.

transfer rates and corresponding sheet temperature variations. This information is necessary for fixing the streamwise length of the processing station.

The objective of this paper is to provide a universal solution for the streamwise temperature variation in a moving sheet which is situated in either a quiescent or moving fluid. Although the universal result is based on exact solutions of the relevant conservation equations, the streamwise temperature variation can be determined by purely algebraic operations. The algebraic form of the universal result enables its convenient use as a design tool. A further focus of the paper is to assess the errors incurred when an approximate method, the relative-velocity method, is used instead of the exact solutions.

The literature on fluid flow and heat transfer for the moving sheet problem can be conveniently classified into two groupings: (1) heat transfer in the fluid corresponding to a prescribed, artificial temperature boundary condition at the fluid-sheet interface without thermal participation of the sheet, and (2) coupled heat transfer in the fluid and in the moving sheet for the special case of a moving sheet in an otherwise quiescent fluid. With regard to group 1, the thermal boundary condition that was most often employed is the isothermal sheet [1–6]. In [7], the difference between the surface temperature and the fluid temperature was assumed to vary as a power of the streamwise coordinate. This prescribed temperature was chosen in order to obtain a similarity solution for the temperature field.

The papers of group 2 [8–13] represent, for the most part, numerical solutions of the coupled energy equations for the moving sheet and induced boundary layer flow in the otherwise quiescent fluid. In [13], an integral method is employed while in [11], a simplified model was used in which a known heat transfer coefficient was imposed on the moving sheet. Although the papers of group 2 dealt with the conjugate problem in some form, all except [11,13] required the solution of differential equations.

In contrast to the aforementioned papers, the present work deals directly with the variation of the moving-sheet temperature in response to heat exchange with an adjacent fluid which may either be quiescent or moving in its own right. The treatment of the sheet temperature as controlled by the first law of thermodynamics is in contrast to the models employed in group 1, and the involvement of an independently moving fluid includes the models of group 2 as a special case. Another unique feature of the present work is that, from the standpoint of a user, only algebraic operations are needed to obtain the streamwise temperature variation of the sheet.

A potential shortcut to solving the problem in which both participating media move independently is to use the *relative-velocity model*. According to this approach, the velocity of the slower medium is subtracted from that of the faster medium, yielding the relative velocity, U_{rel}. Then, the heat transfer formulas for the faster medium are evaluated using the relative velocity as input. In numerous heat transfer textbooks [15–19], the use of the relative-velocity model is implicitly suggested.

2. Establishment of a knowledge base

The purpose of this section of the paper is to establish a knowledge base to support the algebraic universal solutions to be developed in the next part of the paper. The knowledge base will consist of heat transfer information at the interface between the moving fluid and the moving sheet. This information will be obtained for certain convenient thermal boundary conditions which will later be generalized to accommodate any streamwise variation of the interface temperature. Since the thermal knowledge base necessarily depends on the solution of the velocity problem, the relevant velocity knowledge base will be presented first.

2.1. Velocity information

The solution to the velocity problem that is needed as a prerequisite for solving the heat transfer problem is already available in the literature [14]. The velocity problem in the fluid corresponding to the simultaneous motion of the sheet and the fluid yields a boundary layer similarity solution in which the similarity variable η is defined as

$$\eta = y\sqrt{\frac{U_{\mathrm{rel}}}{\nu x}} \tag{1}$$

where $U_{\mathrm{rel}} = |U_\infty - U_{\mathrm{s}}|$. The x and y coordinates which appear in Eq. (1) are illustrated in Figs. 1 and 2. The corresponding streamwise and transverse velocities, u and v, respectively, are given by

$$u = U_{\mathrm{rel}} f', \quad v = \frac{1}{2}\sqrt{\frac{\nu U_{\mathrm{rel}}}{x}}(\eta f' - f) \tag{2}$$

in which $' = \mathrm{d}/\mathrm{d}\eta$. The quantities f and f' are solutions of a Blasius-type equation

$$f''' + \frac{1}{2} f'' f' = 0 \tag{3}$$

However, the boundary conditions for Eq. (3) differ from those that accompany the classical Blasius equation. The change in the boundary conditions stems from the need to accommodate both the sheet velocity U_{s} at $y = 0$ and the freestream fluid velocity U_∞ as $y \to \infty$. These boundary conditions, plus the impermeability condition at $y = 0$, transform to

$$f(0) = 0, f'(0) = \frac{U_{\mathrm{s}}}{U_{\mathrm{rel}}} \quad \text{and} \quad f'(\eta \to \infty) = \frac{U_\infty}{U_{\mathrm{rel}}} \tag{4}$$

From the definition of $U_{\mathrm{rel}} = |U_\infty - U_{\mathrm{s}}|$, the velocity ratios appearing in Eq. (4) can be rewritten as

$$\frac{U_{\mathrm{s}}}{U_{\mathrm{rel}}} = \frac{1}{\left|1 - \frac{U_\infty}{U_{\mathrm{s}}}\right|}, \quad \frac{U_\infty}{U_{\mathrm{rel}}} = \frac{1}{\left|1 - \frac{U_{\mathrm{s}}}{U_\infty}\right|} \tag{5}$$

Therefore, the velocity problem is parameterized by the ratio of U_∞/U_{s}.

For future reference, it is useful to list the values of $f''(0)$ as a function of U_∞/U_{s}. The special role of $f''(0)$ is its starting value in a numerical marching procedure which begins at $\eta = 0$ and proceeds to $\eta \to \infty$. A physical interpretation of $f''(0)$ is the dimensionless wall shear

$$\frac{\tau_{\mathrm{wall}}}{\frac{1}{2}\rho U_{\mathrm{rel}}^2} = \frac{2|f''(0)|}{\sqrt{\frac{U_{\mathrm{rel}} x}{\nu}}} \tag{6}$$

A listing of the $f''(0)$ values is given in Table 1.

Inspection of Table 1 reveals that the direction of the wall shear depends on whether $U_\infty > U_{\mathrm{s}}$ or $U_{\mathrm{s}} > U_\infty$. In the former case, the fluid drags the sheet and, in turn, in the sheet tends to retard the motion of the fluid. Consequently, the shear imposed by the sheet on the flowing fluid acts in the negative x direction, which accounts for the minus signs appearing in Table 1. On the other hand, the case of the faster moving sheet yields a force which the sheet exerts on the fluid in the positive x direction, with corresponding plus signs in the table.

2.2. Heat transfer information

Suppose that the temperature at the interface, $y = 0$, between the flowing fluid and the sheet can be described by the second-degree polynomial

$$T_{\mathrm{f}}(x, 0) - T_\infty = a_0 + a_1 x + a_2 x^2 \tag{7}$$

where a_0, a_1, and a_2 are constants. At this early point of the development, attention is focused on the fluid and

Table 1
Listing of $f''(\eta = 0)$ values

$\frac{U_\infty}{U_s}$	$f''(\eta = 0)$
0.000	−0.4439
0.100	−0.4832
0.200	−0.5279
0.300	−0.5797
0.400	−0.6421
0.500	−0.7204
0.600	−0.8238
0.700	−0.9713
0.800	−1.214
0.900	−1.717
1.111	1.726
1.250	1.177
1.429	0.9257
1.667	0.7689
2.000	0.6573
2.500	0.5704
3.333	0.4991
5.000	0.4377
10.00	0.3829
∞	0.3319

the heat transfer processes in the sheet are not being considered. Those processes will be the focus of the next section of the paper.

It is proposed to solve for the temperature distribution in the fluid subject to the boundary condition Eq. (7). To this end, it is tentatively postulated that

$$T_f(x,y) - T_\infty = a_0\Theta_0(\eta) + a_1\Theta_1(\eta)x + a_2\Theta_2(\eta)x^2 \tag{8}$$

where η is the similarity variable defined by Eq. (1). In order that Eq. (8) satisfies the boundary condition of Eq. (7), it is necessary that

$$\Theta_0(0) = 1 \quad \Theta_1(0) = 1, \quad \text{and} \quad \Theta_2(0) = 1 \tag{9}$$

It is also necessary to verify that the proposed temperature solution of Eq. (8) satisfies the boundary layer energy equation for the fluid, which is

$$\left(u\frac{\partial T_f}{\partial x} + v\frac{\partial T_f}{\partial y}\right) = \alpha_f\frac{\partial^2 T_f}{\partial y^2} \tag{10}$$

Upon introduction of u and v from Eq. (2) and T_f from Eq. (8), there follows

$$\begin{aligned} &U_{rel}f'\left[a_0\frac{\partial\Theta_0}{\partial x} + a_1\left(\frac{\partial\Theta_1}{\partial x}x + \Theta_1\right) + a_2\left(\frac{\partial\Theta_2}{\partial x}x^2 + 2x\Theta_2\right)\right] \\ &\quad + \frac{1}{2}\sqrt{\frac{\nu U_{rel}}{x}}(\eta f' - f)\left[a_0\frac{\partial\Theta_0}{\partial y} + a_1\frac{\partial\Theta_1}{\partial y}x + a_2\frac{\partial\Theta_2}{\partial y}x^2\right] \\ &\quad = \alpha_f\left[a_0\frac{\partial^2\Theta_0}{\partial y^2} + a_1\frac{\partial^2\Theta_1}{\partial y^2}x + a_2\cdot\frac{\partial^2\Theta_2}{\partial y^2}x^2\right] \end{aligned} \tag{11}$$

which, after elimination of $\partial/\partial x$ and $\partial/\partial y$ in favor of $d/d\eta$ by the chain rule, can be rearranged to read

$$\begin{aligned} &x^{-1}\left[\frac{1}{Pr}\Theta_0'' + \frac{f}{2}\Theta_0'\right] + x^0\left[\frac{1}{Pr}\Theta_1'' + \frac{f}{2}\Theta_1' - f'\Theta_1\right] \\ &\quad + x^1\left[\frac{1}{Pr}\Theta_2'' + \frac{f}{2}\Theta_2' - 2f'\Theta_2\right] = 0 \end{aligned} \tag{12}$$

Since the equality expressed by Eq. (12) must hold for all values of x, it is necessary that each of the bracketed quantities be independently equal to zero. As a consequence,

$$\Theta_0'' + \frac{f}{2}Pr\,\Theta_0' = 0 \tag{13}$$

$$\Theta_1'' + \frac{f}{2}Pr\,\Theta_1' - f'\,Pr\,\Theta_1 = 0 \tag{14}$$

$$\Theta_2' + \frac{f}{2}Pr\,\Theta_2' - 2f'Pr\,\Theta_2 = 0 \tag{15}$$

These equations are, respectively, the governing differential equations for Θ_0, Θ_1, and Θ_2.

The boundary conditions for these equations at the $y = 0$ interface of the fluid and the sheet have already been derived and expressed in Eq. (9). Since the differential equations at issue are of second order, an additional boundary condition is required. That condition can be obtained by examining the behavior of the temperature solution as $y \to \infty$ ($\eta \to \infty$). Since the fluid temperature in the freestream is T_∞, it is necessary that

$$\begin{aligned} T_f(x, y\to\infty) - T_\infty &= a_0\Theta_0(\eta\to\infty) + a_1\Theta_1(\eta\to\infty)x \\ &\quad + a_2\Theta_2(\eta\to\infty)x^2 = 0 \end{aligned} \tag{16}$$

Since, again, the equality must hold for all values of x, it follows that

$$a_0\Theta_0(\eta\to\infty) = 0 \tag{17}$$

$$a_1\Theta_1(\eta\to\infty)x = 0 \tag{18}$$

$$a_2\Theta_2(\eta\to\infty)x^2 = 0 \tag{19}$$

If a_0, a_1, and a_2 are not zero, Eqs. (17)–(19) are satisfied by

$$\Theta_0(\infty) = \Theta_1(\infty) = \Theta_2(\infty) = 0 \tag{20}$$

If Eqs. (13)–(15) are taken together with Eqs. (9) and (17)–(20), the Θ_0, Θ_1, and Θ_2 problems are totally specified. The solutions for these variables depend on the prescribed values of two parameters: the Prandtl number Pr and the velocity ratio U_∞/U_s. Numerical solutions of these equations are readily carried out by utilizing a forward-integration method in which values of $\Theta_i'(0)$ are tentatively assigned and systematically varied until the boundary conditions expressed by Eq. (20) are fulfilled.

The values of $\Theta_i'(0)$ are not only critical for the numerical solution of the governing equations for the Θ_i functions, but they are also closely related to the rate of heat transfer that crosses the interface between the sheet and the adjacent fluid. Therefore, the values of $\Theta_i'(0)$ represent an important part of the knowledge base that is needed for the generalized solutions for the streamwise sheet-temperature variations. It is appropriate to list the values of $\Theta_i'(0)$ for convenient subsequent use. Such a listing is provided in Table 2 for 180 cases covering the entire range of velocity ratios and for Prandtl numbers of 0.7, 5, and 10. This choice of the Prandtl number reflects the use of both air and water as common fluids related to the processing of moving sheets.

In order to facilitate interpolation among the $\Theta_i'(0)$ values for the three selected Prandtl numbers, a scaling of $\Theta_i'(0)$ by the factor $Pr^{0.45}$ has been introduced which effectively narrows the range over which interpolation has to be performed.

Inspection of the table reveals that the values of $\Theta_i'(0)$ are greater at higher Prandtl numbers. This is because the thermal boundary layer becomes thinner as the Prandtl number increases, and, as a consequence, the temperature functions $\Theta_i(\eta)$ must decrease more rapidly between their terminal values of 1 and 0. Also noteworthy is that the sensitivity of $\Theta_i'(0)$ to the velocity ratio U_∞/U_s increases with increasing Prandtl number.

The presentation of the information for $\Theta_i'(0)$ brings the needed knowledge base to completion.

3. Generalized method for the streamwise variation of the sheet temperature

3.1. Heat transfer rate at the fluid-sheet interface

A necessary ingredient in the development of a method for determining the streamwise variation of the sheet temperature is a knowledge of the heat flux, q_w, passing *into* the sheet at the fluid-sheet interface. To this end, Fourier's law is applied to the temperature solution, Eq. (8), at $y = 0$, with the result

$$q_w = k_f \frac{\partial T_f}{\partial y}\bigg|_{y=0} = k_f \frac{\partial(a_0\Theta_0(\eta) + a_1\Theta_1(\eta)x + a_2\Theta_2(\eta)x^2)}{\partial y}\bigg|_{y=0} \tag{21}$$

With the recognition that $\partial\Theta/\partial y = \mathrm{d}\Theta/\mathrm{d}\eta \cdot \partial\eta/\partial y$, Eq. (21) becomes

$$q_w = k_f\sqrt{\frac{U_{rel}}{\nu x}} \cdot \left(a_0\Theta_0'(0) + a_1\Theta_1'(0)x + a_2\Theta_2'(0)x^2\right) \tag{22}$$

The $\Theta_i'(0)$ values needed for the evaluation of q_w are available from Table 2.

3.2. Energy balance for the moving sheet

The generalized method for determining the streamwise variation of the temperature of the moving sheet is based on the first law of thermodynamics. Almost

Table 2
Listing of $\Theta_i'(0)/Pr^{0.45}$ as a function of U_∞/U_s and Pr

$\frac{U_\infty}{U_s}$	$\Theta_0'(0)$			$\Theta_1'(0)$			$\Theta_2'(0)$		
	0.7	5	10	0.7	5	10	0.7	5	10
0.00	−0.410	−0.559	−0.596	−0.943	−1.164	−1.224	−1.316	−1.573	−1.647
0.10	−0.453	−0.594	−0.631	−1.014	−1.233	−1.294	−1.407	−1.662	−1.739
0.20	−0.500	−0.635	−0.673	−1.097	−1.313	−1.376	−1.510	−1.769	−1.848
0.30	−0.554	−0.685	−0.723	−1.194	−1.410	−1.476	−1.633	−1.897	−1.980
0.40	−0.618	−0.746	−0.786	−1.311	−1.532	−1.599	−1.786	−2.057	−2.144
0.50	−0.696	−0.825	−0.866	−1.459	−1.685	−1.758	−1.980	−2.261	−2.354
0.60	−0.800	−0.931	−0.974	−1.657	−1.893	−1.972	−2.238	−2.537	−2.639
0.70	−0.946	−1.085	−1.132	−1.942	−2.197	−2.285	−2.612	−2.941	−3.055
0.80	−1.186	−1.342	−1.395	−2.412	−2.705	−2.809	−3.235	−3.616	−3.752
0.90	−1.715	−1.915	−1.987	−3.459	−3.846	−3.986	−4.625	−5.134	−5.321
1.11	−1.700	−1.931	−1.911	−3.373	−3.688	−3.812	−4.484	−4.913	−5.078
1.25	−1.164	−1.299	−1.286	−2.288	−2.478	−2.556	−3.032	−3.294	−3.400
1.43	−0.917	−1.004	−0.994	−1.786	−1.912	−1.966	−2.356	−2.535	−2.612
1.67	−0.766	−0.818	−0.809	−1.471	−1.552	−1.592	−1.933	−2.054	−2.110
2.00	−0.656	−0.683	−0.676	−1.245	−1.291	−1.317	−1.626	−1.701	−1.741
2.50	−0.572	−0.576	−0.569	−1.068	−1.082	−1.098	−1.384	−1.419	−1.445
3.33	−0.504	−0.487	−0.480	−0.920	−0.905	−0.910	−1.182	−1.179	−1.191
5.00	−0.445	−0.410	−0.401	−0.792	−0.747	−0.741	−1.005	−0.962	−0.960
10.0	−0.392	−0.343	−0.328	−0.675	−0.599	−0.580	−0.842	−1.241	−0.737
∞	−0.344	−0.279	−0.258	−0.564	−0.452	−0.417	−0.686	−0.548	−0.506

without exception, the moving sheets encountered in practice are sufficiently thin so as to obviate the need to consider temperature variations across their thickness. On this basis, the sheet temperature may be treated as a function only of the streamwise coordinate x. In addition, the thin-sheet model reduces the axial conduction of heat in the sheet to a negligible value. In this light, the energy balance on the moving sheet can be focused on an element whose streamwise length is $\mathrm{d}x$, whose thickness is t, and whose spanwise width is W. The energy balance brings together the rate of heat transfer to the element with the rate of internal energy increase for the mass contained within the element. The rate at which mass passes through the element is $\dot{m}$. For a non-stretchable sheet, $\dot{m}$ can be expressed as

$$\dot{m} = \rho_s W t U_s \tag{23}$$

The surface area across which heat flows into the element is $jW\mathrm{d}x$, where $j = 1$ or 2, depending, respectively, on whether there is heat flow on one or both of the faces of the sheet. With these notations, the energy balance becomes

$$\rho_s W t U_s \cdot c_s \cdot \mathrm{d}T_s = q_w \cdot jW\,\mathrm{d}x \tag{24}$$

Rearrangement of Eq. (24) and subsequent introduction of the expression for q_w from Eq. (22) yields

$$\begin{aligned}\frac{\mathrm{d}T_s}{\mathrm{d}x} &= \frac{q_w \cdot j}{\rho_s t \cdot c_s \cdot U_s} \\ &= \frac{jk_f}{\rho_s t \cdot c_s \cdot U_s}\sqrt{\frac{U_{rel}}{\nu x}} \\ &\quad \cdot \left(a_0\Theta_0'(0) + a_1\Theta_1'(0)x + a_2\Theta_2'(0)x^2\right)\end{aligned} \tag{25}$$

To obtain a dimensionless representation, it is convenient to introduce a dimensionless streamwise coordinate χ defined by

$$\chi = \left(\frac{U_{rel} \cdot k_f \rho_f c_f}{(\rho_s c_s t U_s)^2}\right) \cdot x \tag{26}$$

Upon substitution of this transformation into the energy balance of Eq. (25), there emerges

$$\frac{\mathrm{d}T_s}{\mathrm{d}\chi} = \frac{1}{\sqrt{Pr}}\frac{j}{\sqrt{\chi}}\left(b_0\Theta_0'(0) + b_1\Theta_1'(0)\chi + b_2\Theta_2'(0)\chi^2\right) \tag{27}$$

Inspection of Eq. (27) indicates that the transformation has accomplished its purpose and has reduced the energy balance to dimensionless form. The b_0, b_1, and b_2 are a new set of arbitrary constants to replace the initial arbitrary set a_0, a_1, and a_2. After rearrangement of Eq. (27) and subsequent integration between 0 and χ, there results

$$\int_{T_s(0)}^{T_s} \mathrm{d}T_s = \frac{j}{\sqrt{Pr}}\int_0^{\chi} \frac{\left(b_0\Theta_0'(0) + b_1\Theta_1'(0)\lambda + b_2\Theta_2'(0)\lambda^2\right)}{\sqrt{\lambda}}\mathrm{d}\lambda \tag{28}$$

When the actual integration is performed, the sheet temperature variation takes the form

$$T_s(\chi) = T_s(0) + \frac{j}{\sqrt{Pr}}\left(\frac{b_0\Theta_0'(0)\chi^{0.5}}{0.5} + \frac{b_1\Theta_1'(0)\chi^{1.5}}{1.5} + \frac{b_2\Theta_2'(0)\chi^{2.5}}{2.5}\right) \tag{29}$$

Rearrangement of Eq. (29) leads to the dimensionless form

$$\frac{T_s(\chi) - T_\infty}{T_s(0) - T_\infty} = 1 + \frac{j}{\sqrt{Pr}}\left(\frac{1}{T_s(0) - T_\infty}\right)\left(\frac{b_0\Theta_0'(0)\chi^{0.5}}{0.5} + \frac{b_1\Theta_1'(0)\chi^{1.5}}{1.5} + \frac{b_2\Theta_2'(0)\chi^{2.5}}{2.5}\right) \tag{30}$$

It is convenient to define

$$c_i = \frac{b_i}{T_s(0) - T_\infty} \tag{31}$$

so that

$$\frac{T_s(\chi) - T_\infty}{T_s(0) - T_\infty} = 1 + \frac{j}{\sqrt{Pr}}\left(\frac{c_0\Theta_0'(0)\chi^{0.5}}{0.5} + \frac{c_1\Theta_1'(0)\chi^{1.5}}{1.5} + \frac{c_2\Theta_2'(0)\chi^{2.5}}{2.5}\right) \tag{32}$$

Although Eq. (32) expresses T_s as a function of χ, it is by no means a completed solution in as much as c_0, c_1 and c_2 remain to be determined. The completion of the solution will be dealt with in the next section of the paper.

3.3. Method for obtaining the completed solution for $T_s(\chi)$

To begin the development of the method for completing the solution for $T_s(\chi)$, the quantities c_1 and c_2 are temporarily set equal to zero and c_0 is taken equal to one. This choice of c_0 does not in any way affect the final solution of the problem. Under these conditions, Eq. (32) becomes

$$\frac{T_s^{\mathrm{I}}(\chi) - T_\infty}{T_s(0) - T_\infty} = 1 + \frac{j}{\sqrt{Pr}}\left(\frac{\Theta_0'(0)\chi^{0.5}}{0.5}\right) \tag{33}$$

Eq. (33) represents a first estimate for $T_s(\chi)$. To reflect this fact, the superscript 'I' has been appended.

It now remains to refine this initial estimate of the temperature variation, $T_s(\chi)$. To this end, $(T_s^{\mathrm{I}}(\chi) - T_\infty)/(T_s(0) - T_\infty)$ is fitted with a second-degree polynomial in χ. Therefore,

$$1 + \frac{j}{\sqrt{Pr}}\left(\frac{\Theta_0'(0)\chi^{0.5}}{0.5}\right) \cong \left(c_0^{\mathrm{I}} + c_1^{\mathrm{I}}\chi + c_2^{\mathrm{I}}\chi^2\right) \tag{34}$$

Inspection of the left-hand side of Eq. (34) indicates that all of the constants that appear there are known. The Prandtl number is part of the specification of the problem, and the value of $\Theta_0'(0)$ is listed in Table 2 as a function of the both the velocity ratio U_∞/U_s and the Prandtl number. The quantity j is either 1 or 2 depending on whether the heat transfer between the sheet and the fluid respectively occurs at one or both of the faces of the sheet. Since the left-hand side is a known function of χ, the coefficients c_0^{I}, c_1^{I}, and c_2^{I} can be routinely determined by making use of the least-squares capability of a mathematical package such as Excel.

The next step in the method for completing the solution for the sheet temperature is to make use of Eq. (32) and specialize it to the presently known values of c_0^{I}, c_1^{I}, and c_2^{I}, so that

$$\frac{T_s^{\mathrm{II}}(\chi)-T_\infty}{T_s(0)-T_\infty}=1+\frac{j}{\sqrt{Pr}}\left(\frac{c_0^{\mathrm{I}}\Theta_0'(0)\chi^{0.5}}{0.5}+\frac{c_1^{\mathrm{I}}\Theta_1'(0)\chi^{1.5}}{1.5}+\frac{c_2^{\mathrm{I}}\Theta_2'(0)\chi^{2.5}}{2.5}\right) \tag{35}$$

At this stage, it is useful to compare $(T_s^{\mathrm{I}}(\chi)-T_\infty)/(T_s(0)-T_\infty)$ and $(T_s^{\mathrm{II}}(\chi)-T_\infty)/(T_s(0)-T_\infty)$. Such a comparison, conveniently made graphically, will indicate whether or not it is necessary to continue the refinement procedure or to accept $(T_s^{\mathrm{II}}(\chi)-T_\infty)/(T_s(0)-T_\infty)$ as the completed solution. It is the authors' experience that further levels of refinement will be required.

The proceed to the next level, the $(T_s^{\mathrm{II}}(\chi)-T_\infty)/(T_s(0)-T_\infty)$ variation conveyed by Eq. (35) is fitted with another second-degree polynomial in χ, so that

$$1+\frac{j}{\sqrt{Pr}}\left(\frac{c_0^{\mathrm{I}}\Theta_0'(0)\chi^{0.5}}{0.5}+\frac{c_1^{\mathrm{I}}\Theta_1'(0)\chi^{1.5}}{1.5}+\frac{c_2^{\mathrm{I}}\Theta_2'(0)\chi^{2.5}}{2.5}\right)\cong\left(c_0^{\mathrm{II}}+c_1^{\mathrm{II}}\chi+c_2^{\mathrm{II}}\chi^2\right) \tag{36}$$

Once again, all of the constants that appear on the left-hand side are known, thereby enabling the coefficients c_0^{II}, c_1^{II}, and c_2^{II} to be readily determined by means of a least-squares fit. With these values, the next level of refinement of $(T_s^{\mathrm{III}}(\chi)-T_\infty)/(T_s(0)-T_\infty)$ can be written as

$$\frac{T_s^{\mathrm{III}}(\chi)-T_\infty}{T_s(0)-T_\infty}=1+\frac{j}{\sqrt{Pr}}\left(\frac{c_0^{\mathrm{II}}\Theta_0'(0)\chi^{0.5}}{0.5}+\frac{c_1^{\mathrm{II}}\Theta_1'(0)\chi^{1.5}}{1.5}+\frac{c_2^{\mathrm{II}}\Theta_2'(0)\chi^{2.5}}{2.5}\right) \tag{37}$$

Further appraisal of the progress of the solution can be made by means of a graphical comparison of $(T_s^{\mathrm{II}}(\chi)-T_\infty)/(T_s(0)-T_\infty)$ and $(T_s^{\mathrm{III}}(\chi)-T_\infty)/(T_s(0)-T_\infty)$. The outcome of such a comparison is expected to indicate a trend toward convergence. To continue toward the attainment of complete convergence, the refinement process can be repeated to obtain $(T_s^{IV}(\chi)-T_\infty)/(T_s(0)-T_\infty)$, $(T_s^{V}(\chi)-T_\infty)/(T_s(0)-T_\infty)$ and so on, until the differences between successive iterants is sufficiently small.

4. Universal nature of the solution

It was stated in Section 1 that the solutions to be set forth here for the streamwise variation of the temperature of the moving sheet are universal with respect to most of the parameters that govern the problem. By inspection of Eq. (37), it can be seen that the solution does not depend on any material property of the sheet, its velocity, and its thickness. Furthermore, the solution is independent of the fluid properties aside from the Prandtl number. The velocities of the sheet and the fluid need not be specified as such, but only their ratio is required. Therefore, the only parameters which affect the solution for the sheet temperature are the Prandtl number and the ratio U_∞/U_s.

Another key feature of the solution method is its guaranteed convergence regardless of the initial iterant. The authors have experimented with several initial iterants and have demonstrated conclusively that the converged solution is independent of the starting distribution of $T_s^{\mathrm{I}}(\chi)$. As a further test of the inherent tendency of the procedure to converge, the authors artificially introduced errors in one or more of the iterants. It was observed that even in these disturbed cases, convergence was achieved to the same final result as was obtained when no disturbances were introduced.

5. Illustration of the use of the method

It is relevant to demonstrate the application of the method and, specifically, to show the manner in which a converged solution is obtained. For the demonstration, the case $U_\infty/U_s = 5$ and $Pr = 0.7$ was selected. As indicated in the previous section, these are the only parametric values that are needed for the solution.

To begin, Eq. (33) is used along with the value $\Theta_0'(0) = -0.379$ from Table 2, which corresponds to $U_\infty/U_s = 5$ and $Pr = 0.7$. In addition, j was selected equal to 2. With these inputs, Eq. (33) becomes

$$\frac{T_s^{\mathrm{I}}(\chi)-T_\infty}{T_s(0)-T_\infty}=1-\frac{2}{\sqrt{0.7}}\left(\frac{0.379\chi^{0.5}}{0.5}\right)=1-1.812\chi^{0.5}\cong\left(c_0^{\mathrm{I}}+c_1^{\mathrm{I}}\chi+c_2^{\mathrm{I}}\chi^2\right) \tag{38}$$

The next step involves obtaining the values of the c coefficients from a least-squares fit, with the results $c_0^{\mathrm{I}} = 0.8968$, $c_1^{\mathrm{I}} = -6.224$, and $c_2^{\mathrm{I}} = 14.162$.

Next, these c values are introduced into Eq. (35) along with the $\Theta_i'(0)$ values from Table 2, from which there emerges

$$\frac{T_s^{\mathrm{II}}(\chi)-T_\infty}{T_s(0)-T_\infty}=1-1.63\chi^{0.5}+6.69\chi^{1.5}-11.6\chi^{2.5} \tag{39}$$

which corresponds to the conclusion of the second iteration. The next iteration involves a least-squares fit of

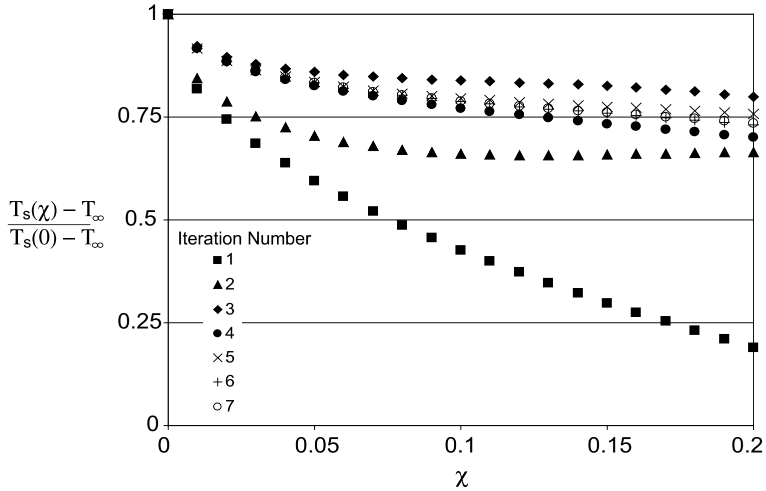

Fig. 3. Representative pattern of convergence of the iterative procedure which yields the variation of the sheet temperature in the streamwise direction ($U_\infty/U_s = 5$, $Pr = 0.7$).

the right-hand side of Eq. (39). The constants which result from that fit are: $c_0^{\mathrm{II}} = 0.8936$, $c_1^{\mathrm{II}} = -3.90$, and $c_2^{\mathrm{II}} = 14.66$. With these constants, the equation for the third iteration is

$$\frac{T_s^{\mathrm{III}}(\chi) - T_\infty}{T_s(0) - T_\infty} = 1 - 1.62\chi^{0.5} + 4.19\chi^{1.5} - 12.0\chi^{2.5} \tag{40}$$

This procedure is continued until convergence.

To identify the attainment of convergence, it is convenient to plot the individual iterations and to examine their behavior. This has been performed in Fig. 3. It is seen from the figure that convergence is fully achieved after seven iterations. The pattern of convergence is marked by successive decreasing overshoots and undershoots with respect to the converged result.

6. Errors incurred by the use of the relative-velocity model

As noted in the Introduction, the use of the relative-velocity model is implied in a number of heat transfer textbooks which have already been referenced in the Introduction. In those texts, problems involving moving sheets were presented without any of the background knowledge required for the solution of those problems. The only provided information that could be construed as being applicable is that for the classical Blasius flat plate problem in which the plate is stationary and the fluid moves over it. Therefore, in the absence of other information, it may be inferred that the authors' intended that these moving-sheet problems be solved using the results for a moving fluid and a stationary sheet.

A natural extension of this inference is the relative-velocity model. In such a model, when both the sheet and the fluid move independently, the smaller of the velocities is subtracted from the larger, and the difference is denoted as the relative velocity, U_{rel}. Then, the slower-moving medium is regarded as non-moving while the faster-moving medium is treated as if its velocity is U_{rel}. Next, the skin friction and heat transfer formulas are identified for the case in which only one of the media moves. These formulas are evaluated by replacing the velocity of the moving medium with U_{rel}. In practice, the faster of the two media is that to which the identified formulas belong.

6.1. The case of $U_\infty > U_s$

Consider, for concreteness, the situation in which the velocity of the fluid, U_∞, exceeds that of the sheet, U_s. Correspondingly, the relative velocity would be equal to $U_{\mathrm{rel}} = U_\infty - U_s$. For this model, the similarity variable is as given by Eq. (1), and the solution for the velocity field is imbedded in Eqs. (2) and (3). However, a change must be made in the boundary conditions relative to those stated in Eq. (4). In particular, the appropriate physical boundary conditions are: $u = 0$ and $v = 0$ at $y = 0$, and $u = U_{\mathrm{rel}}$ as $y \to \infty$. In terms of the transformed variables, these boundary conditions become

$$f(0) = 0, \quad f'(0) = 0 \quad \text{and} \quad f'(\eta \to \infty) = 1 \tag{41}$$

The wall shear stress corresponding to this model may be derived from its definition

$$\tau_{\mathrm{wall,RVM}} = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu\sqrt{\frac{U_{\mathrm{rel}}}{\nu x}} f''_{\mathrm{RVM}}(0) \tag{42}$$

The abbreviation *RVM* indicates the *relative-velocity model*. Eq. (42) can be rearranged in dimensionless form as

$$\frac{\tau_{\text{wall,RVM}}}{\frac{1}{2}\rho_f U_{\text{rel}}^2} = \frac{2|f''_{\text{RVM}}(0)|}{\sqrt{Re_x}} \tag{43}$$

in which $Re_x = U_{\text{rel}}x/\nu$.

It is useful to compare the wall shear results from the relative-velocity model with those obtained from the complete model which takes account of the independent motions of the two media. The shear stress results from the complete model are expressed in Eq. (6). When ratio of the shear stresses from the two models is formed, there results

$$\frac{\tau_{\text{wall,RVM}}}{\tau_{\text{wall}}} = \frac{|f''_{\text{RVM}}(0)|}{|f''(0)|} \tag{44}$$

This ratio can be readily evaluated by making use of Table 1. The numerator of Eq. (44) is equal to 0.3319 which is read from the bottom line of the table.

Inspection of the values listed in Table 1 in conjunction with Eq. (44) reveals that the largest errors in the wall shear are encountered for those cases in which the velocities of the two media are virtually equal.

The main focus of the accuracy assessment of the relative-velocity model is the streamwise variation of the sheet temperature. The scheme by which the sheet temperature was determined for the complete model in which both media are in motion will now be modified to apply to the relative-velocity model. The modification involves the reinterpretation of $\Theta_0'(0)$, $\Theta_1'(0)$, and $\Theta_2'(0)$. For the relative-velocity model, the numerical values of these quantities are read from the bottom line of Table 2, that is, for $U_\infty/U_s = \infty$. Aside from this change, the iterative scheme that was discussed earlier continues to apply.

To demonstrate specific results, consideration will be given to two cases: (a) $U_\infty/U_s = 5$, $Pr = 0.7$ and (b) $U_\infty/U_s = 1.43$, $Pr = 0.7$. These cases were selected to illustrate the nature of the comparison for a large value of U_∞/U_s and for a value of U_∞/U_s close to 1. The results for case (a) are illustrated in Fig. 4. The figure shows two curves, with the open symbols representing the solution to the exact model and the filled symbols corresponding to the relative-velocity model. The abscissa is the dimensionless streamwise coordinate χ defined by Eq. (26). The error due to the use of the relative-velocity model is seen to grow with increasing downstream distance, reaching a value of 25% at $\chi = 0.2$.

For case (b), the results are conveyed in Fig. 5. From this figure, it is clear that larger errors are in evidence when the velocity ratio U_∞/U_s differs only slightly from one. For example, at $\chi = 0.1$, the error associated with the relative-velocity model is approximately 50%.

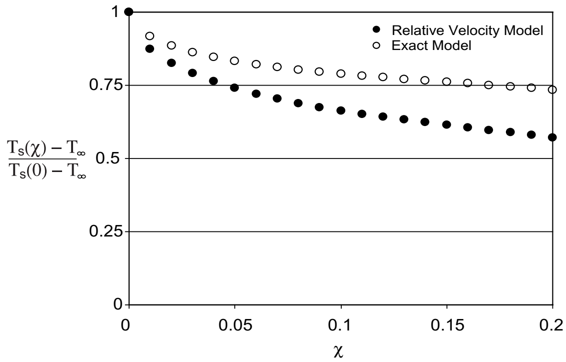

Fig. 4. Comparison of the streamwise variations of the sheet temperature predicted by the relative-velocity model and by the exact solution for $U_\infty/U_s = 5$, $Pr = 0.7$.

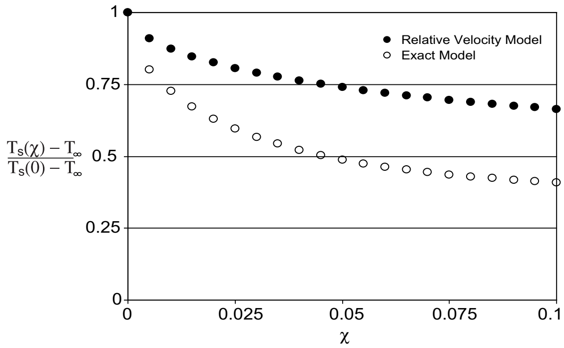

Fig. 5. Comparison of the streamwise variations of the sheet temperature predicted by the relative-velocity model and the exact solution for $U_\infty/U_s = 1.43$, $Pr = 0.7$.

6.2. The case of $U_s > U_\infty$

The treatment of the case where $U_s > U_\infty$ follows the same pattern as that already described in detail for the case in which $U_\infty > U_s$. In fact, there is only one substantive change in the analysis. That change is to use the boundary conditions $f(0) = 0$, $f'(0) = 1$, $f'(\infty) = 0$ when $U_s > U_\infty$ to replace $f(0) = 0$, $f'(0) = 0$, $f'(\infty) = 1$ which were used for $U_\infty > U_s$. The definition of U_{rel} is $U_s - U_\infty$ when $U_s > U_\infty$. Aside from these changes, the discussion presented for $U_\infty > U_s$ applies equally well for $U_s > U_\infty$.

7. Concluding remarks

A method for determining universal solutions for the streamwise variation of the temperature of a moving sheet in the presence of an independently moving fluid has been developed. The first step in the development of the method was to generate an extensive knowledge base obtained from solutions of the boundary layer differential equations for the dual-motion situation. These solutions covered a large number of cases parameterized by two operating conditions: (a) the ratio of the fluid velocity to the sheet velocity and (b) the Prandtl number. The velocity ratio ranged from the case in which the fluid was stationary in the presence of a moving sheet to the case in which a flowing fluid passed over a non-moving sheet. The Prandtl numbers selected for study reflected the practical use of air and water as common fluids for the processing of moving sheets. The database consisted of two parts: (a) normal derivatives of the velocity at the interface of the sheet and the fluid and (b) normal derivatives of the temperature at the same location. For the temperature problem, provision was made to accommodate streamwise variations of the temperature of the moving sheet. These temperature variations were not prescribed in advance. Rather, they resulted from the dynamic interaction of the conjugate heat transfer processes.

Once the database was generated, it was employed to obtain universal solutions for the streamwise variation of the temperature of the moving sheet. The method for obtaining these temperature results is purely algebraic. It is an iterative process which requires no more than a least-squares curve-fitting capability. It was demonstrated that the successive iterations rapidly converge to a final result. Furthermore, the final converged result was shown to be independent of the initial iterant. In addition, the iterative process is self correcting in the presence of an inadvertent error.

In addition to the exact solutions which were obtained as described in the foregoing, approximate solutions were obtained by the so-called *relative-velocity model*. This model is based on the use of the relative velocity between the moving media. In that model, equations valid for a situation in which only one of the media move are adapted to the two-moving-media case by introduction of the relative velocity in those equations. Worked-out examples demonstrated that the errors incurred by the use of the relative-velocity model are greatest when the individual velocities of the two media are very close to being equal.

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